Bifurcations and Fractal Zeta Functions of Orbits
University of Zagreb

Energy forms on random and stretched fractals

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Motivation: Analysis on fractals

Plan of the lecture

1. (Self–similar) fractals

2. Analysis on (self–similar) fractals
The Dirichlet form approach (Kusuoka, Kigami)
Model case: Sierpinski gasket

3. Spectral asymptotics

4. The V−variable model

5. Spectral asymptotics in the V−variable model

6. Stretched fractals (= quantum graphs with vanishing edge lengths)

1. Introduction: Self similar fractals

1.1. Definition and Examples

$K \subseteq \mathbb{R}^n$ is called self similar, if

$$K = \bigcup_{i=1}^{M} S_i(K)$$

where $M \geq 2$ and $S_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ similitudes.

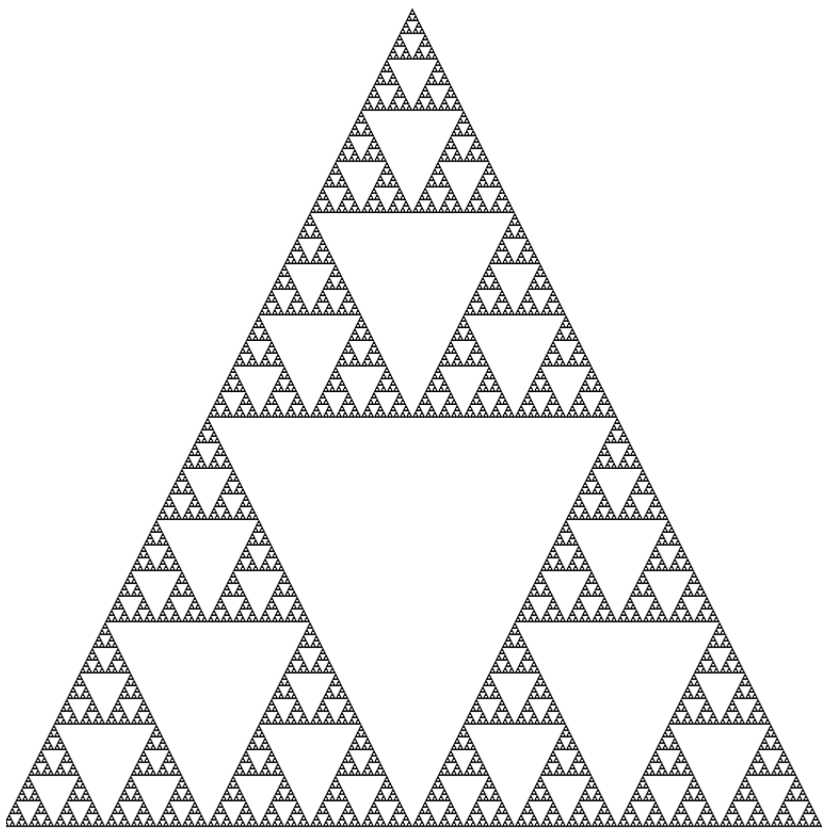
Exp. Sierpinski gasket:

A, B, C vertices of a unilateral triangel

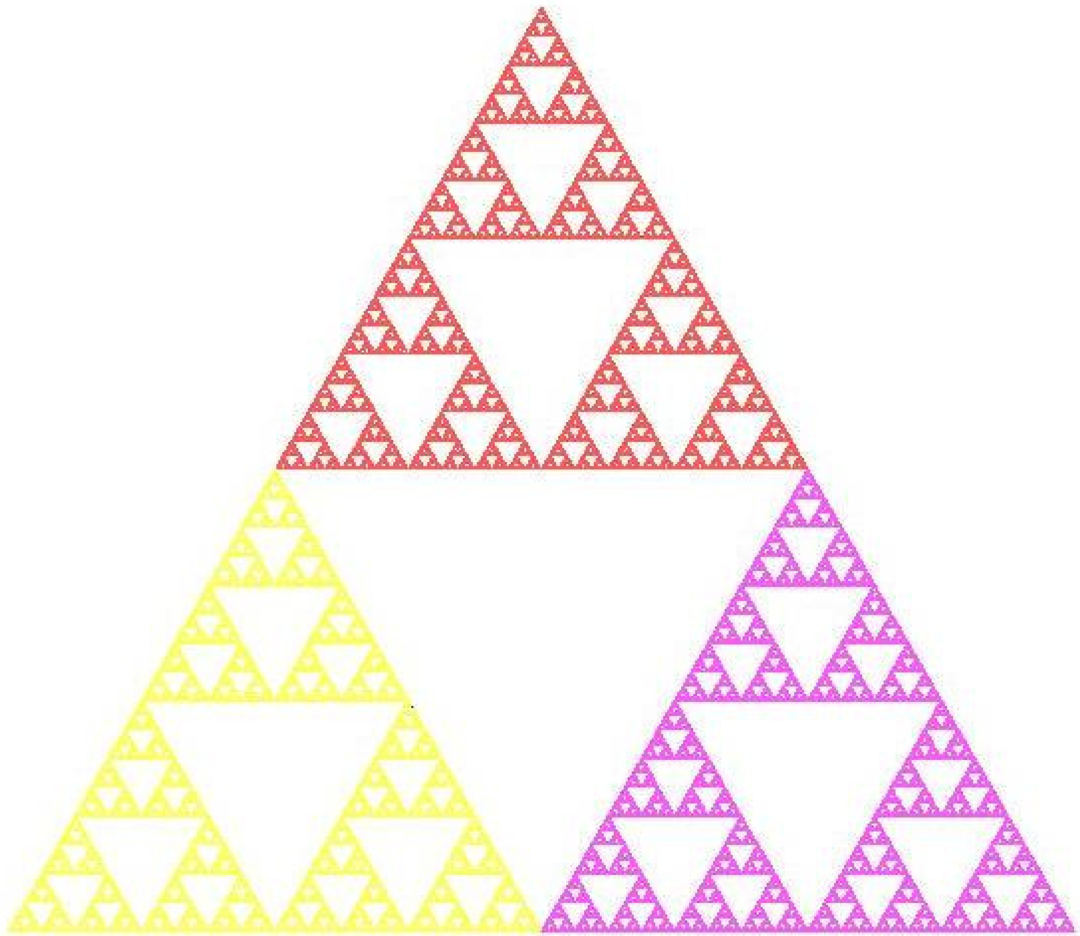
Family $\mathfrak{S} = \{S_1, S_2, S_3\}$ of contractions on $\mathbb{R}^2$, where

$S_1(x) = \frac{1}{2}(x - A) + A, S_2(x) = \frac{1}{2}(x - B) + B, S_3(x) = \frac{1}{2}(x - C) + C$

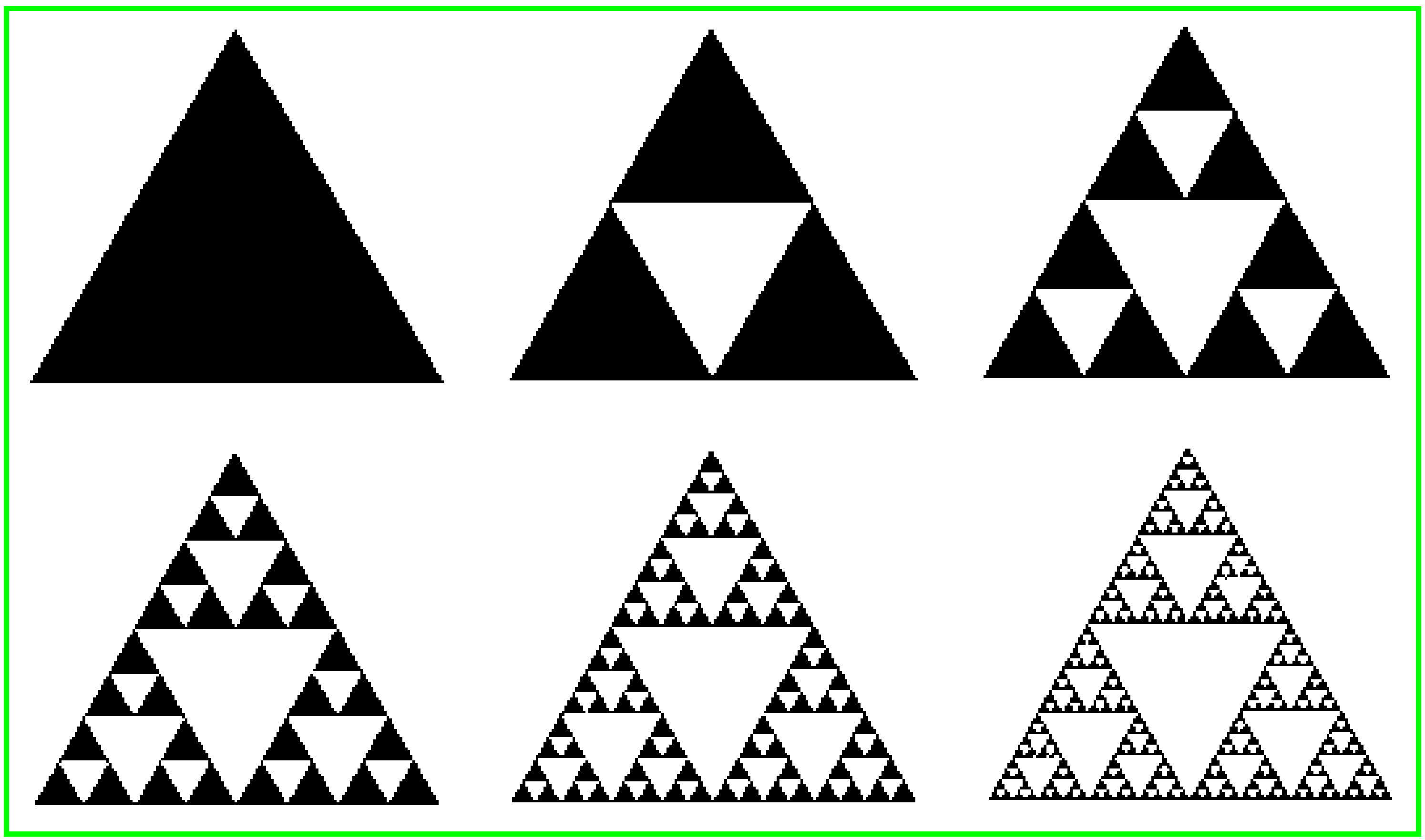
There is a unique (non empty and compact) set K, the so–called Sierpinski gasket:



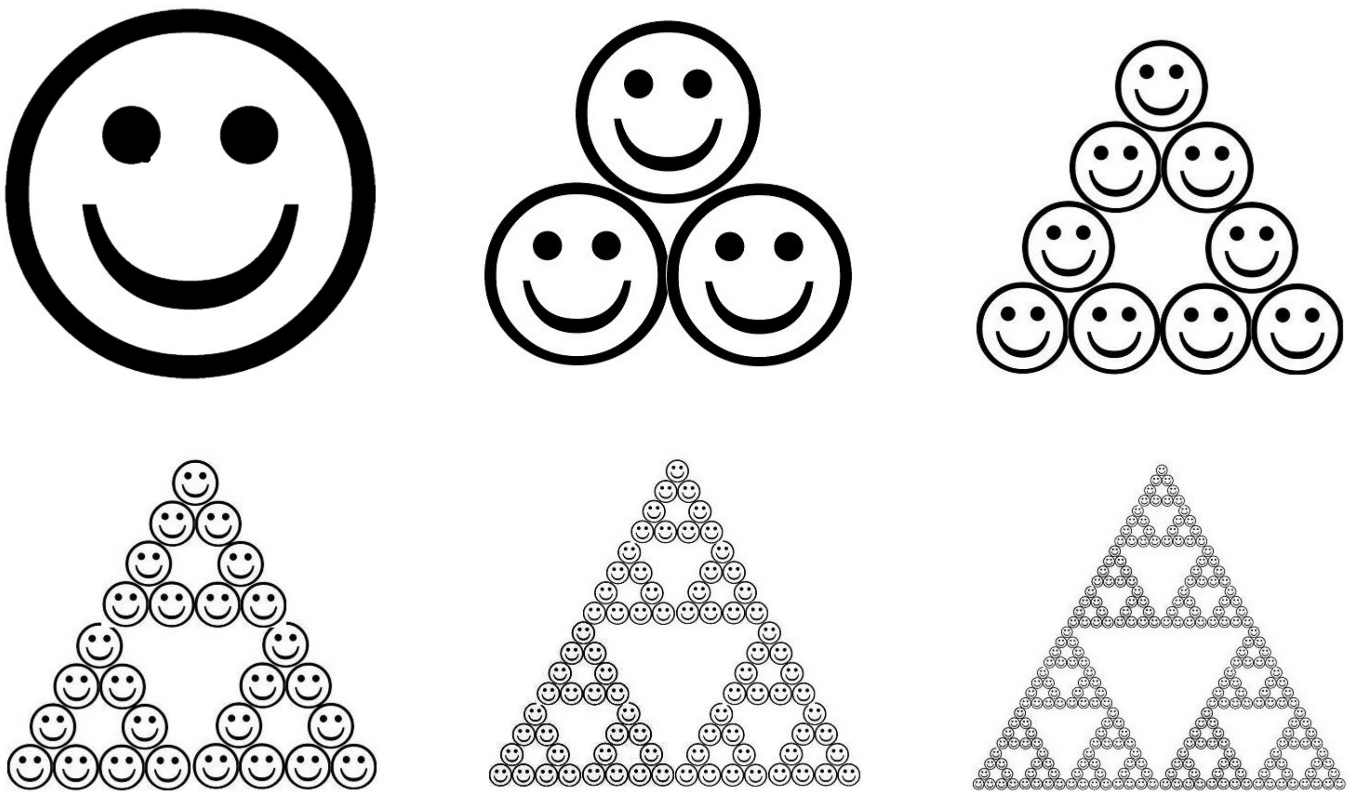
Again:



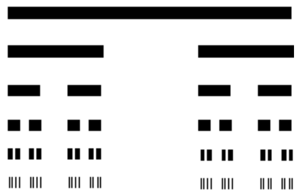
It can be obtained by iteration of the three mappings:



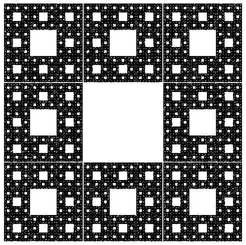
Hereby, you can start with any set:



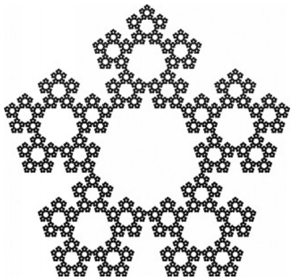
Further examples for self–similarity:



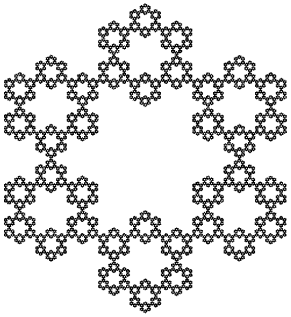
a) Cantor set



b) Sierpinski carpet



c) Pentagasket



d) Snowflake

2. Dirichlet form and Laplacian on the Sierpinski gasket

2.1. Interludium: Analysis on fractals

aim: Definition of the Laplacian Δ
(wave/heat/Poisson/Schrödingier equation)

Problem: Fractals are too „rough“

$\Rightarrow$ no tangent space

$\Rightarrow$ new notion of derivative necessary

Classical approaches:

- limit of difference operators (Dirichlet form theory)
  Kusuoka, Kigami, Lapidus, Mosco, Hambly, Teplyaev, Strichartz,...

- Construction of the „natural“ Brownian motion as the limit of a sequence of appropriate renormalized random walks
  Kusuoka, Barlow, Bass, Perkins, Lindstrøm; Sabot, Metz,...

- Martin boundary theory on the Code space
  Denker, Sato, Koch,...

- (fractal dimensional) traces of function spaces (for exp. Sobolev spaces) or via Riesz potentials
  Triebel, Haroske, Schmeißer,...; Zähle

New approaches:

- Generalized Laplacians (Δ–Beltrami, Hodge–Δ, Dirac–Δ)
  M. Hinz, Teplyaev, Rogers,...

- Non-commutative Geometry: Interpretation of the fractal in terms of spectral triple
  Bellissard, Falconer, Samuel, Lapidus; Cipriani, Guido, Isola, ...

- Theory of resistance forms
  Kigami, Kajino, Alonso–Ruiz, F. ,...

- Approximation by quantum graphs
  Teplyaev, Kelleher, Alonso–Ruiz, F. ...; Mugnolo, Lenz, Keller, Post, Kuchment, ...

2.2. Kusuoka' s approach

Aim: Define Δ_K Laplacian on K

Idea:

- Define „fractal analogue" $\mathcal{E}_K[u]$ of $\mathcal{E}[u] = \int\limits_\Omega |\nabla u|^2 dx$
- $\mathcal{E}_K(u,v) := \frac{1}{2}\left(\mathcal{E}_K[u+v] - \mathcal{E}_K[u] - \mathcal{E}_K[v]\right)$ bilinear form
- Δ_K via Gauß–Green–formula:

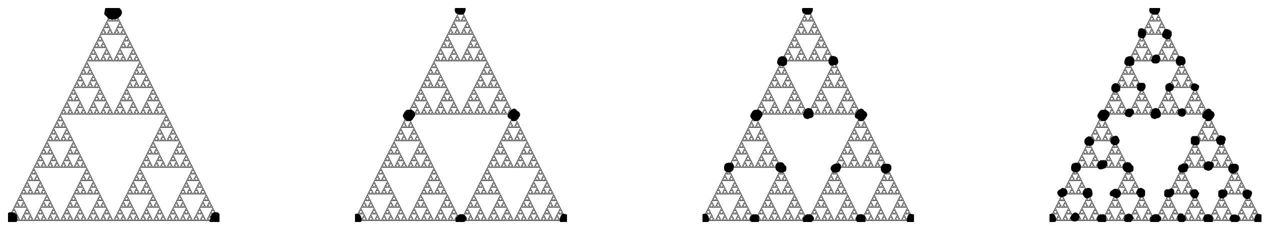
$$\int\limits_K (\Delta_K u)\, v\, d\mu = \text{boundary terms } - \mathcal{E}_K(u,v)$$

(cf. $\int\limits_\Omega \Delta u \cdot v =$ boundary terms $- \int\limits_\Omega \nabla u \cdot \nabla v$)

via: Dirichlet forms on Graphs

Approximation of K:

$$V_0 := \{A, B, C\}, \qquad V_n := \bigcup_{i=1}^{3} S_i(V_{n-1}), n \geq 1$$



V_0, V_1, V_2 and V_3

$$(V_n) \uparrow, \quad V_* := \bigcup_{n \geq 0} V_n = \sup_{n \geq 0} V_n, \quad K = \overline{V_*}$$

Let $u : V_* \longrightarrow \mathbb{R}$

Ansatz: $\mathsf{E}_n[u] := \varrho^n \sum_{p \in V_n} \sum_{|p-q|=2^{-n}} (u(p) - u(q))^2, \qquad n \geq 0.$

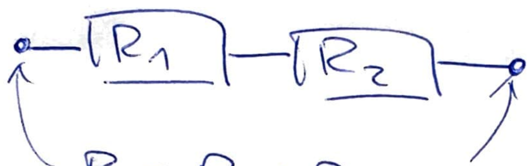
ϱ scaling number (defined and obtained by „Gaussian principle“)

It turns out that $\varrho_F = 5/3$.

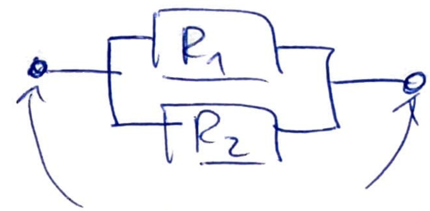
Finding the right conductance renormalization

(1st) way: conductance equivalent networks ($C_{xy} = r_{xy}^{-1}$)

resistances

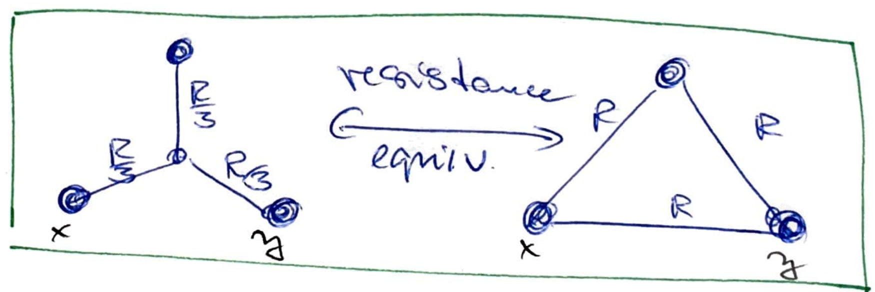


$R = R_1 + R_2$



$R^{-1} = R_1^{-1} + R_2^{-1}$

Claim (Δ-Y-rule)

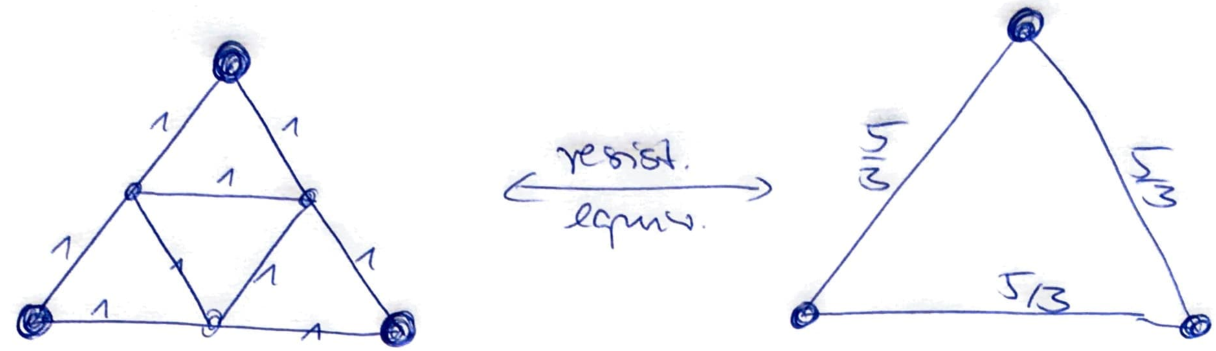


Proof l.h.s. $R_{xy} = \frac{R}{3} + \frac{R}{3} = \frac{2R}{3}$

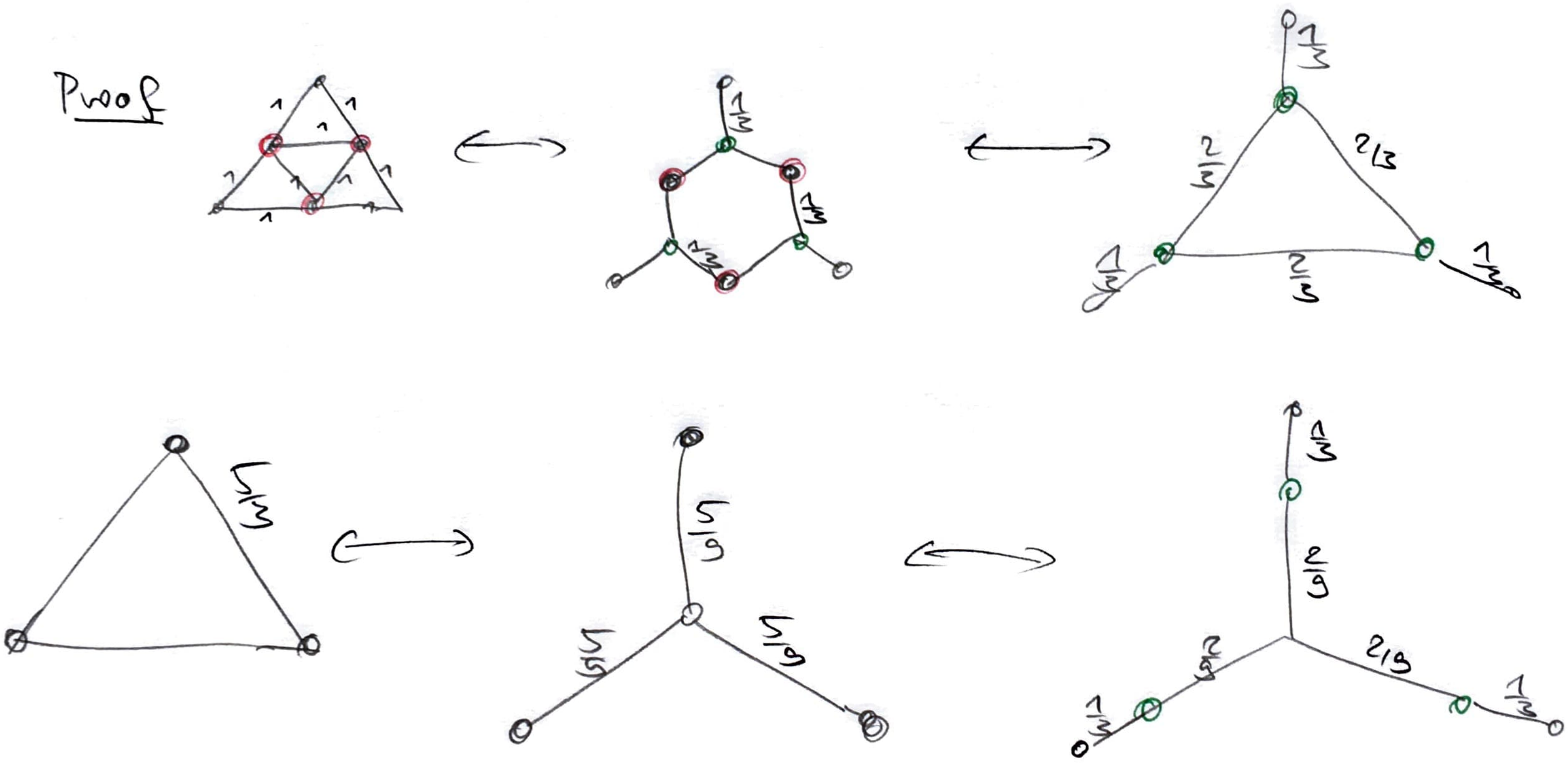
r.h.s $R_{xy}^{-1} = \frac{1}{R} + \frac{1}{2R} = \frac{3}{2R}$ ✓

Remark ∃ Δ-Y-rule for unbalanced case, much more complicated!

Corollary

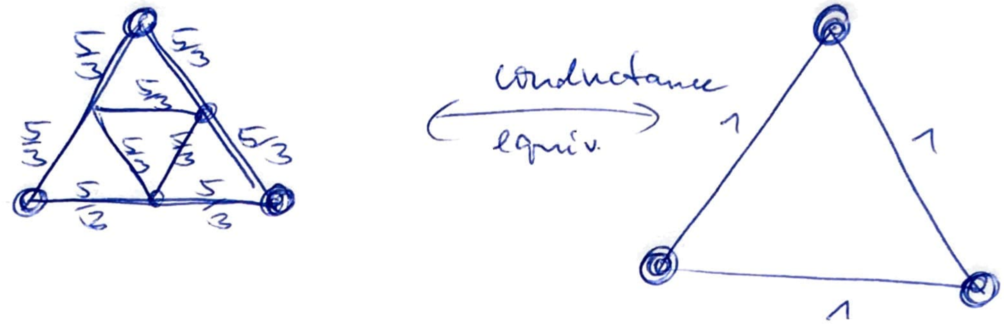


Proof



Corollary

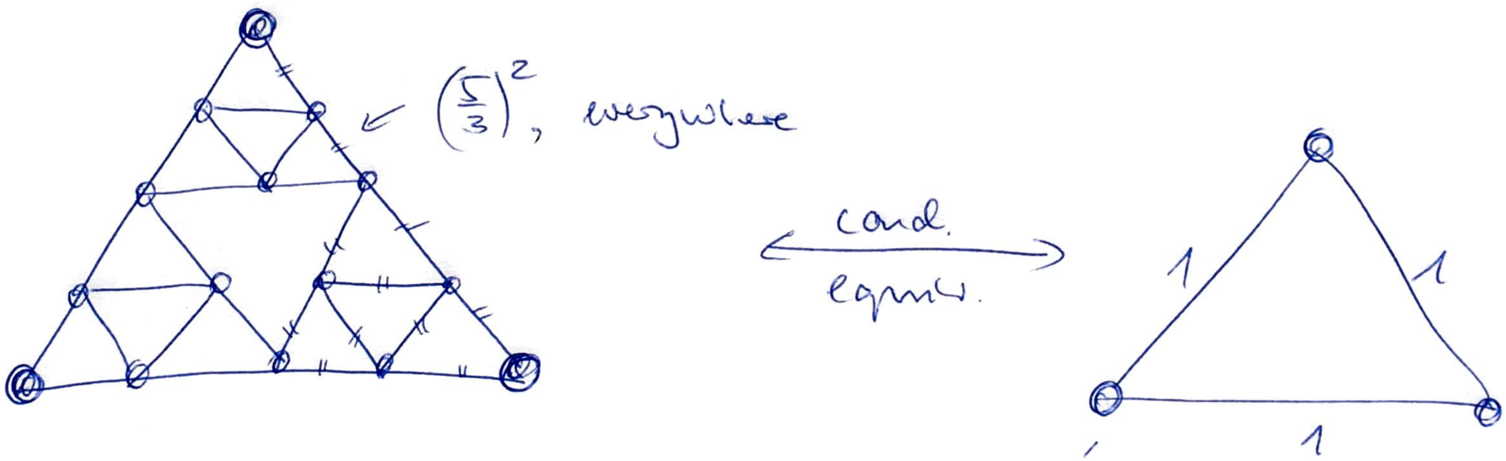
i)



$\Rightarrow \rho = \frac{5}{3}$

By induction, it follows that:

ii)



thanks to → finite ramification

& → self-similarity

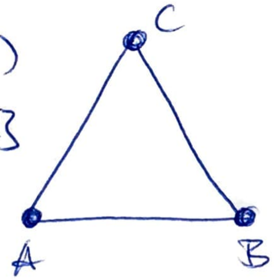
$\Rightarrow \rho_n = \left(\frac{5}{3}\right)^n$ ⊠

2nd way (in finding g): seeking for harmonic extensions "Gaußian principle"

from graph Γ_0 to Γ_1

$\Gamma_0 = (V_0, E_0)$

$V_0 = \{A, B, C\}$



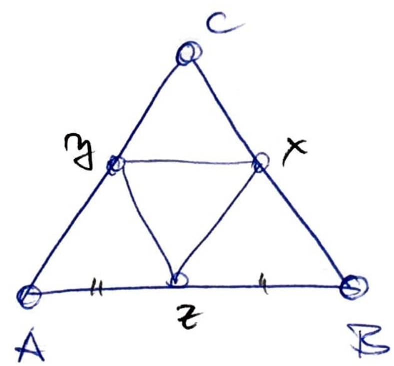
$u : V_0 \longrightarrow \mathbb{R}$, $u(A) =: u_A$ ....

$$\mathcal{E}_0[u] = (u_A - u_B)^2 + (u_B - u_C)^2 + (u_A - u_C)^2$$

now: $\Gamma_1 = (V_1, E_1)$

$V_1 = \{A, B, C, x, y, z\}$

$V_0 \subseteq V_1$



Seeking for $u : V_1 \to \mathbb{R}$ s.t. $u_{|V_0}$ as above s.t.

$$\mathcal{E}_1[u] \longrightarrow \min$$

where

$$\mathcal{E}_1[u] = (u_A - u(z))^2 + (u(z) - u_B)^2 + \ldots \quad (9 \text{ summands})$$

This argmin $\mathcal{E}_1[u]$ is the harmonic extension of $u_{|V_0}$!

Calculation shows: $u^*(x) = \frac{u_A + 2u_B + 2u_C}{5}$, ...

and $\mathcal{E}_1[u]_{min} = 3/5\ \mathcal{E}_0[u]$ ⇝ "correction" factor $= 5/3$

again: self simi & finite ramification ⇝ $(5/3)^n$

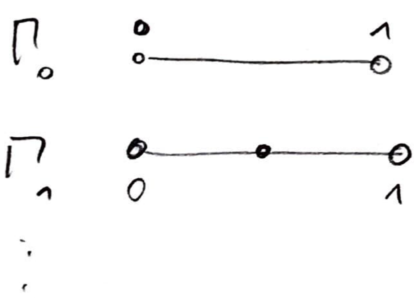
1^{st} & 2^{nd} way give the same result!

Reason: **Voltage is a harmonic function**

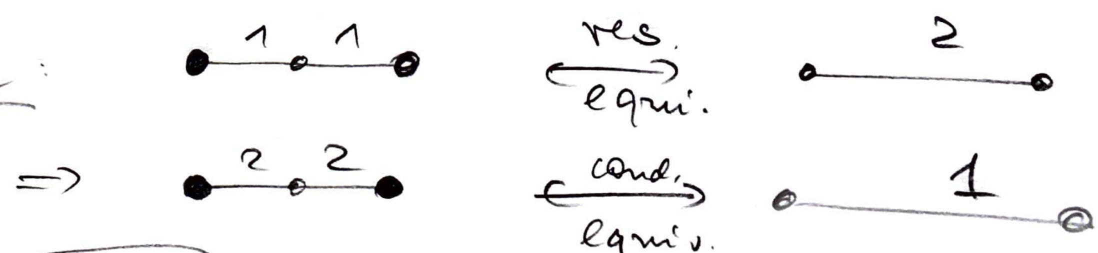
↑ Doyle & Snell: "Random walks + Electric networks"

Why is this the right way to think about it?

Exp $[0,1]$ unit interval



1^{st} way:



$\rho = 2$

goal $\mathcal{E}[u] = \int_0^1 |u'(x)|^2 \, dx$

Kusuokas ansatz: $$\mathcal{E}_n[u] = \sum_{\substack{|x-y|=2^{-n} \\ x,y \in \mathcal{D}_n}} 2^n \, (u(p)-u(q))^2$$ (2^n // g^n)

where $\mathcal{D}_n = \left\{ \frac{x}{2^n} : x = 0 \dots 2^n \right\}$

$$\mathcal{E}_n[u] = \sum_{x \in \mathcal{D}_n \setminus \{1\}} 2^n \left[u\left(x + \frac{1}{2^n}\right) - u(x) \right]^2$$

$$\left[\frac{\left(u(x) + \frac{1}{2^n}\right) - u(x)}{2^{-n}} \right]^2 \cdot \frac{1}{2^n}$$

$\downarrow$

$u'(x)$

" $\longrightarrow$ " $\int_0^1 [u'(x)]^2 \, dx$ (for exp. for $u \in C^1[0,1]$)

↑ Strichartz "PDE on fractals"

Self similarity and finite ramification $\Rightarrow (\mathcal{E}_n[u])_{n\geq 0} \nearrow$

Limit form $\mathcal{E}_K[u] := \lim_{n\to\infty} \mathcal{E}_n[u]$

on $\mathcal{D}_* := \{u : V_* \longrightarrow \mathbb{R} \ : \ \mathcal{E}_K[u] < \infty\}$

Extension from $u \in \mathcal{D}_*$ to $u \in \mathcal{C}(K)$

$\mathcal{D} := \overline{\mathcal{D}_*}$ completion wrt. $\left(||.||^2_{L_2(K,\mu)} + \mathcal{E}_K[.]\right)^{1/2}$

$(\mathcal{E}, \mathcal{D})$ is Dirichlet form on $L_2(K,\mu) \Rightarrow \Delta_K$ Laplace operator

$(\mathcal{E}, \mathcal{D})$ ist regular and local $\Rightarrow$ ex. associated diffusion process $(X_t)_{t\geq 0}$ (Brownian motion on K)

3. Spectral asymptotics in the deterministic model

3.1. Spectral dimension – Definition

Consider $\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ Laplace operator of $\mathbb{R}^n$.

H. Weyl, 1915: The eigenvalue counting function

$$N_n(x) := \#\{\lambda_k \leq x \quad : \quad -\Delta_n u = \lambda_k u \text{ for some } u \neq 0\},$$

(Counting according multiplicities) is well–defined, and for any $n \in \mathbb{N}$ it holds that

$$N_n(x) = (2\pi)^{-n} c_n \mathrm{vol}^n(\Omega) x^{n/2} + o(x^{n/2}), \qquad \text{as} \quad x \to \infty,$$

Hence, $d_S(\Omega) = n$.

3.2. Spectral asymptotics in the self similar deterministic model

Spectral dimension d_S

$$N(x) \sim x^{d_S/2}, \quad x \to \infty.$$

Kigami, Lapidus 1993

$\mu =$ Hausdorff measure on $K \Rightarrow \exists\ C_1, C_2, x_0$ such that
$C_1 x^{d_S/2} \leq N_{D/N}(x) \leq C_2 x^{d_S/2}, \quad x \geq x_0,$

where $\frac{d_S}{2} = \frac{\ln M}{\ln(\varrho M)}$.

Hereby, $M =$ number of copies, $\varrho =$energy scaling factor of $\mathcal{E}$.
We set $T := \varrho M$ (Einstein–Relation!). Hence, $\frac{d_S}{2} = \frac{\ln M}{\ln T}$.

Further remaks:

- **Berry's conjecture, 80's:** There is a Weyl type asymptotics also valid for fractals K, i.e.

$$N_K(x) = c_d \mathcal{H}^d(K) x^{d/2} + o(x^{d/2}), \qquad \text{as} \qquad x \to \infty$$

  with $d := d_H(K)$, c_d a constant independent of K. Is wrong! i.g. $d_H \neq d_S$

- In general we do not have that $\mathcal{E}[u] \preceq \mathcal{H}^d$, i.e. we do not have that
  $\mathcal{E}[u] = \int |\nabla u|^2 d\mathcal{H}^d$

- Second derivatives are easier to define than first derivatives!

4. The $V-$variable model

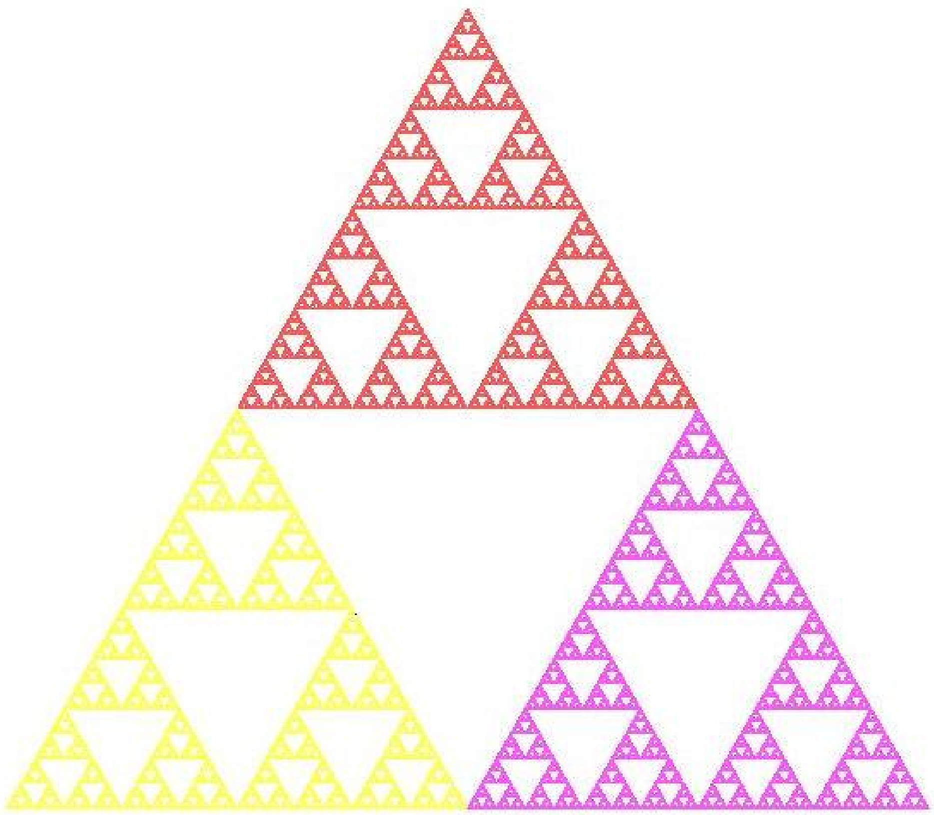
The assumption of strict self similarity can be too restricting (in order to model real world fractal sets).

Way out: „Weakening“ of the assumption, for exp. by

random mixing of different IFS’s

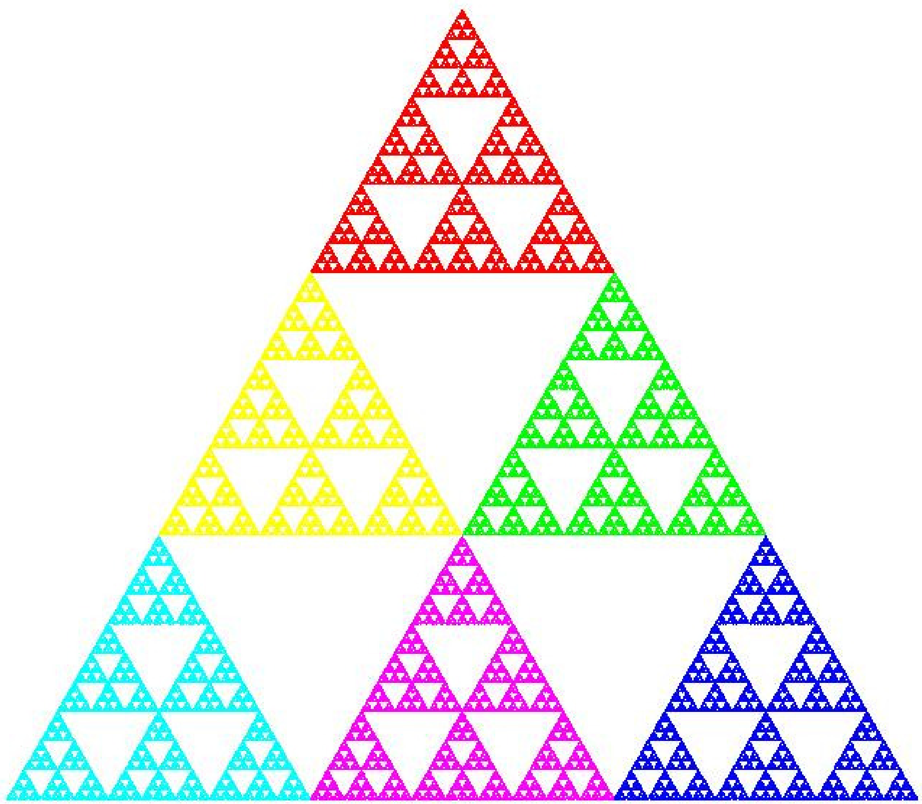
4.1. The model case: SG(2) vs. SG(3)

SG(2)–Sierpinski gasket (F)



measure scaling factor μ
$\mu_F = 1/3$
length scaling factor r
$r_F = 1/2$
energy/resistance scaling factor ϱ $\varrho_F = 5/3$
time scaling factor $T = M\varrho$
$T_F = 5$

SG(3)–modified Sierpinski gasket (G)



measure scaling factor μ

$\mu_G = 1/6$

length scaling factor r

$r_G = 1/3$

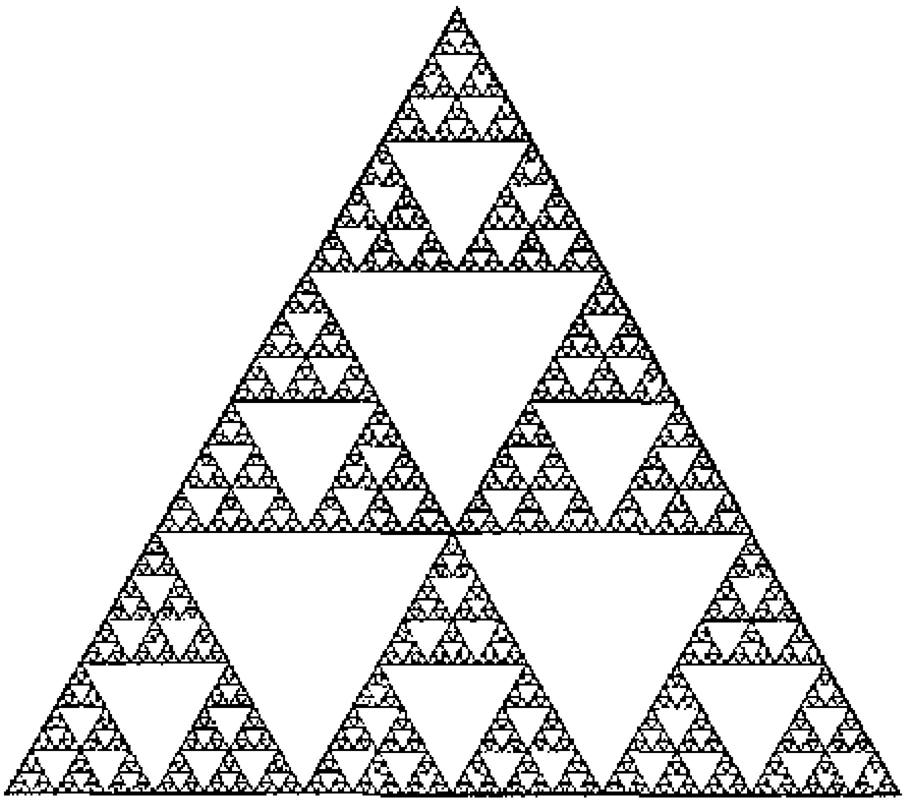
energy/resistance scaling factor ϱ $\varrho_G = 15/7$

time scaling factor $T = M\varrho$

$T_G = 90/7$

Random mixing I

„homogenous random“ ($V = 1$)



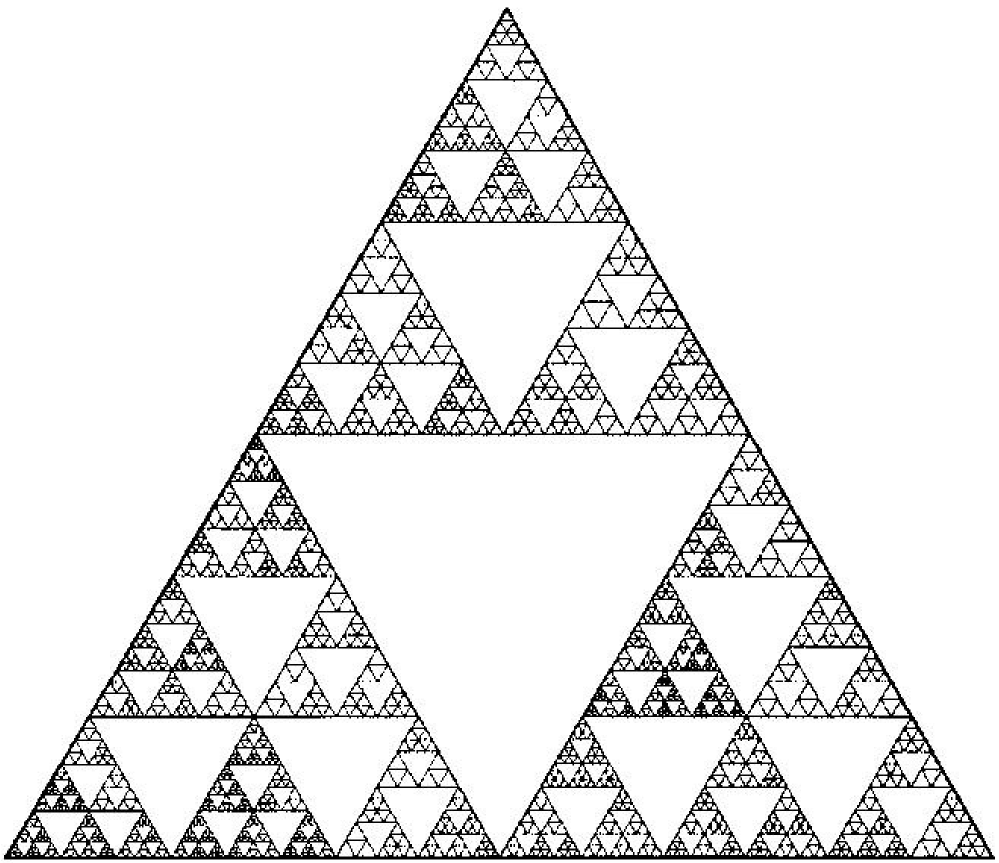
Hambly ('97 BM, '99 heat kernels, '00 Δ)

(Kifer '95, Stenflo '01)

Coding: random sequence

Random mixing II

„random recursive“, „standard random “ $(V = \infty)$



Hambly '92 (BM), Barlow Hambly '97 (Δ)

(Falconer '86, Mauldin Williams '86, Graf '87, Hutchinson Rüschendorf '98, '00)

Coding: random labelled tree

now: „Interpolation" between both the methods I and II
$\rightsquigarrow$ „V–variable fractal" $V \in \mathbb{N} \cup \{\infty\}$

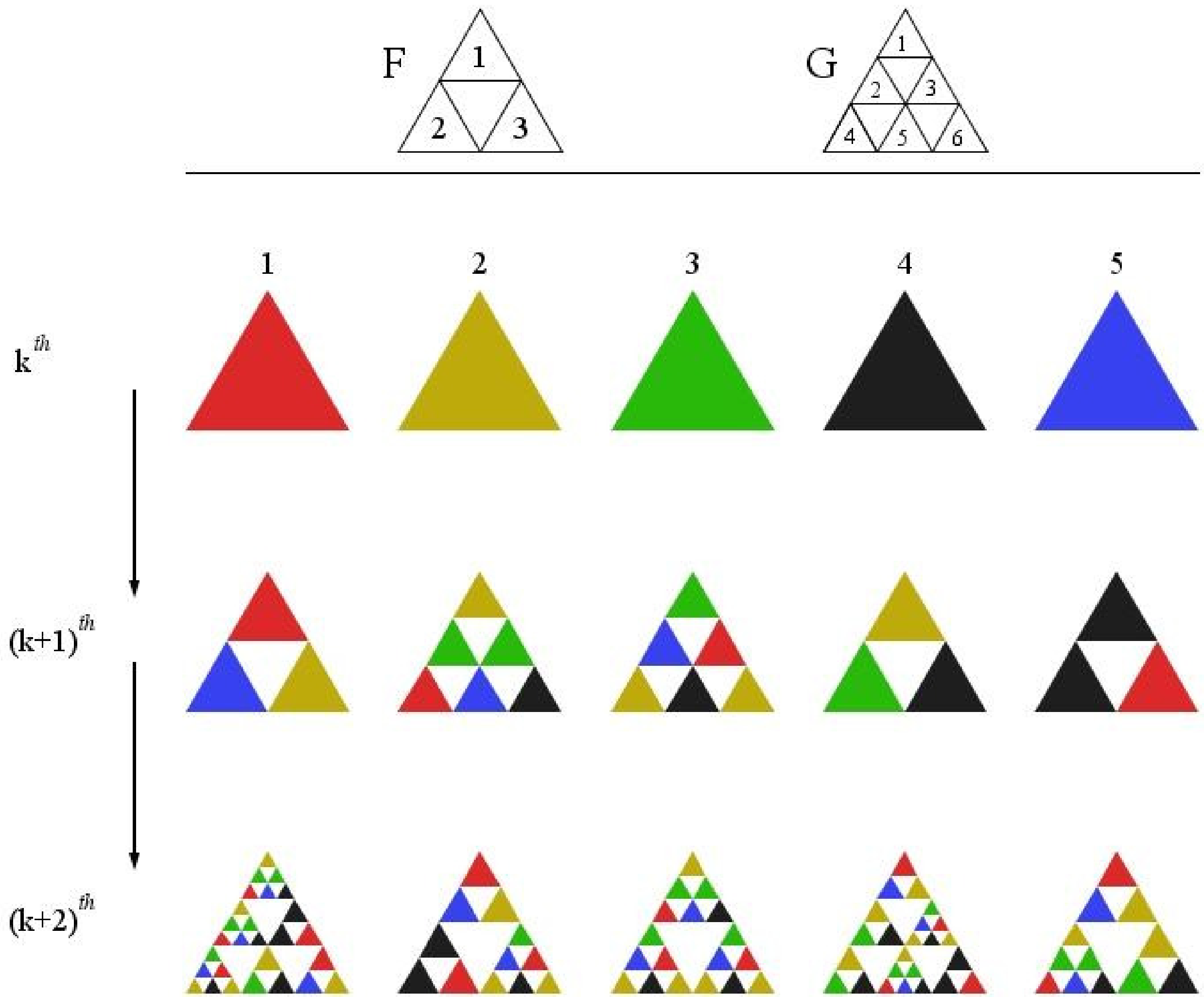
Construction of the V members of the $(k+1)^{st}$ generation from the V members of the k^{th} generation:

1° Choose $\mathcal{F}$ or $\mathcal{G}$ according to probabilities (p_F, p_G).

2° Choose 3 (or 6, resp.) „parents" from generation k for the i^{th} child of generation $k+1$.

Run this loop V times.

We construct V–tuples of sets instead of single sets!

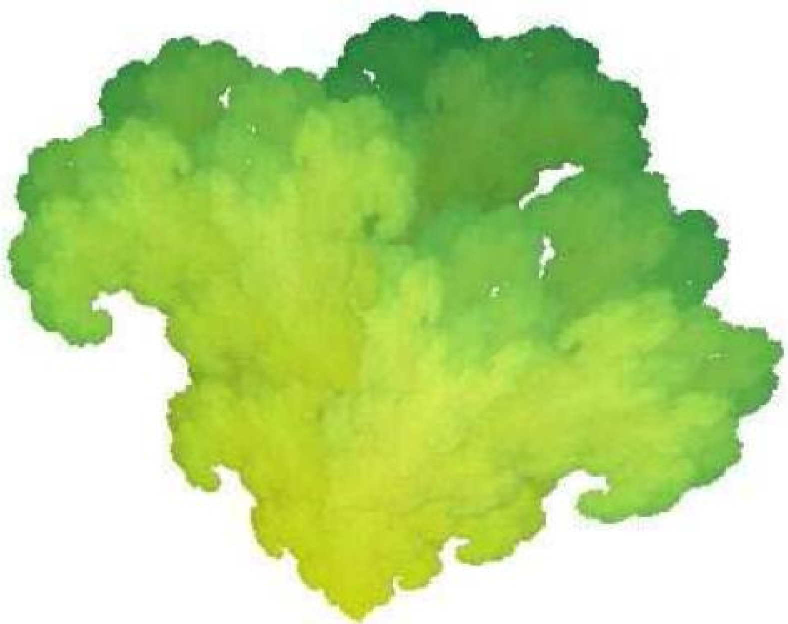

F
1
2
3
G
1
2
3
4
5
6
1
2
3
4
5
k^th
(k+1)^th
(k+2)^th

4.2. Applications

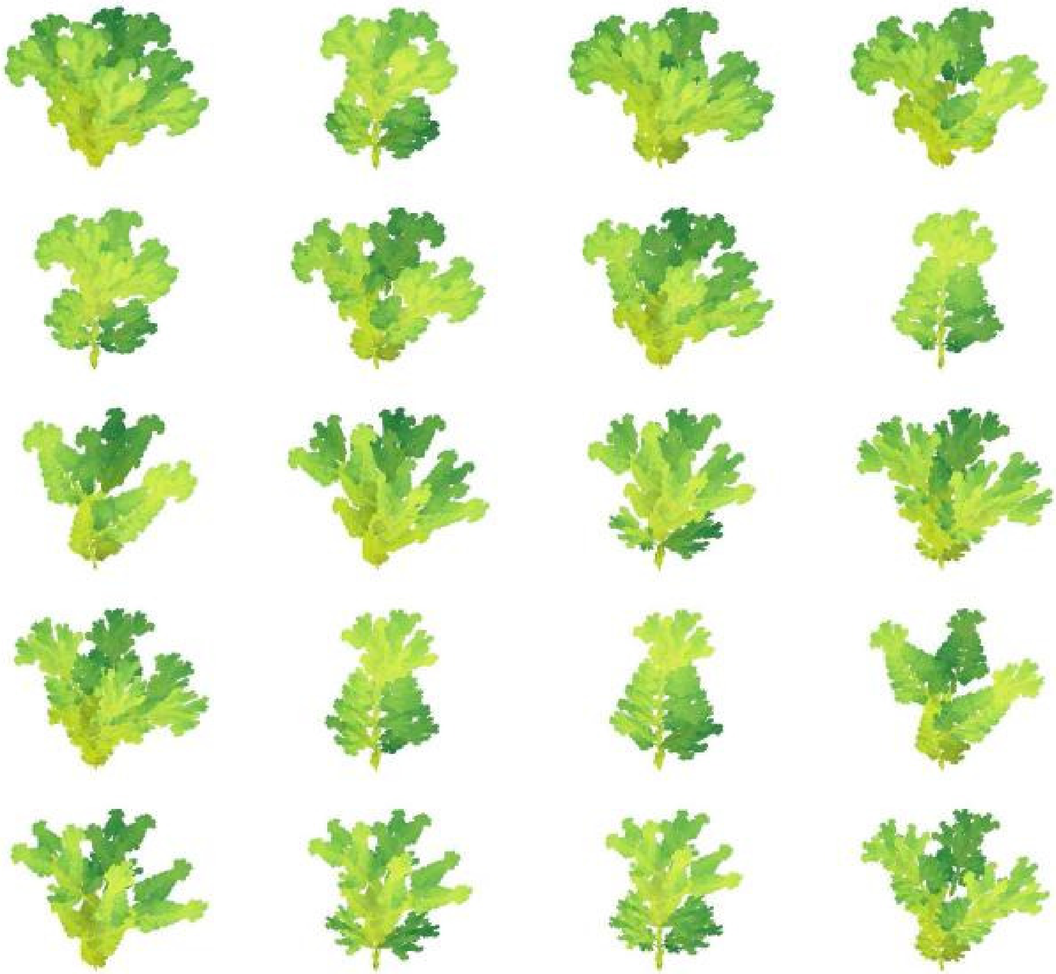
z.B. Modeling of species

- for exp: choose parents wrt. their geographical origin

- „breed“ fractals




salad and fern (attractors of IFS's)



fern–lettuce–hybrids

$V = 2$

5. Spectral asymptotics in the random case

5.1. Construction of the form

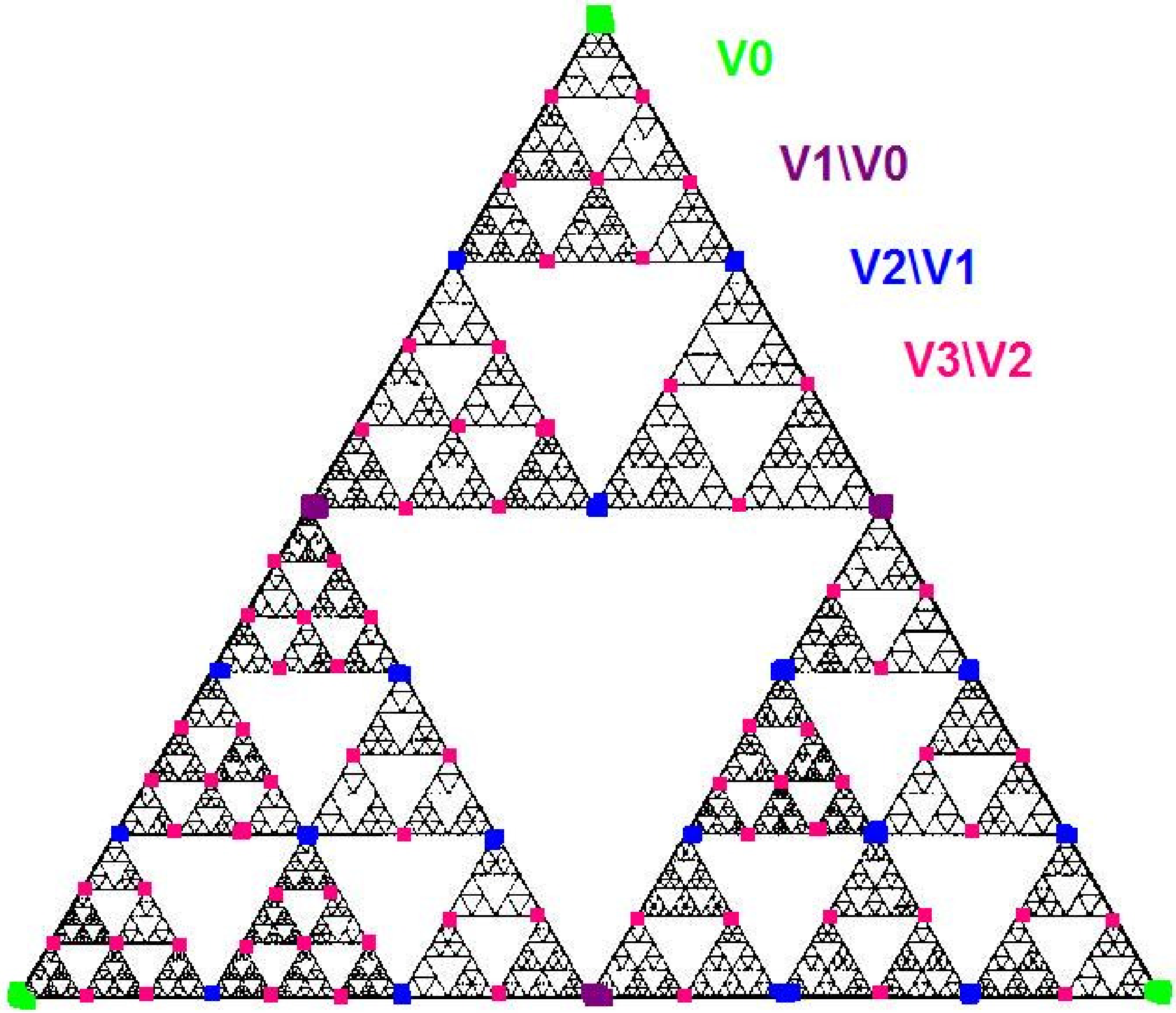
is done hierarchically and ω–wise

(Ω is the set of all V–variable trees)

Define $V_0, V_1, V_2, ...$ ω–wise

Values of a function given on $V_0 \rightsquigarrow$ harmonic extension to $V_1 \setminus V_0$ (i.e. one calculates the function in 3 (or 7) new points)

harmonic extension to $V_2 \setminus V_1$ on EACH of the 3 (or 6) sub triangles of V_1 according to an $\mathcal{F}$– (or $\mathcal{G}$, reps.)–rule


V0
V1\V0
V2\V1
V3\V2

Proceeding like this one obtains a sequence

$$\mathcal{E}_n^{(\omega)}[f] = \sum_{\bar{\imath} \in \omega_n} R(\bar{\imath}) \mathcal{E}_0[f \circ \psi_{\bar{\imath}}]$$

where

$$R(\bar{\imath}) = \prod_{j=1}^{|\bar{\imath}|} \varrho_j, \qquad \varrho_j \in \{\varrho_F = 5/3, \varrho_G = 15/7\}$$

From the construction we obtain:

$$\mathcal{E}_n^{(\omega)}[f_{|V_n}] = \mathrm{inf}\{\mathcal{E}_{n+1}^{(\omega)}[g] : g_{|V_n} = f_{|V_n}\}$$

The limit form $(\mathcal{E}^{(\omega)}, \mathcal{D}(\mathcal{E}^{(\omega)}))$ is a Dirichlet form on $L_2(K(\omega), \mu(\omega))$,

where $K(\omega)$ is the realization of the random set, and $\mu(\omega)$ is a random self similar measure on $K(\omega)$ obtained as the Monge–Kantorovich–limit obtained by applying the Markov–operators $\mathcal{M}_{\mathcal{F}}$ or $\mathcal{M}_{\mathcal{G}}$, reps. according to the tree ω.

5.2. Results

• Homogenous case ($V = 1$): Sequences of F's and G's; Strong law of large numbers, Law of iterated logarithm, martingale theory.

• Recursive case ($V = \infty$): branching theory

• $V-$variable case: we now need products and sums of certain parameters according to the $V-$variable setting.

IDEA: Products of random $V \times V-$matrices coding up the information of the construction process

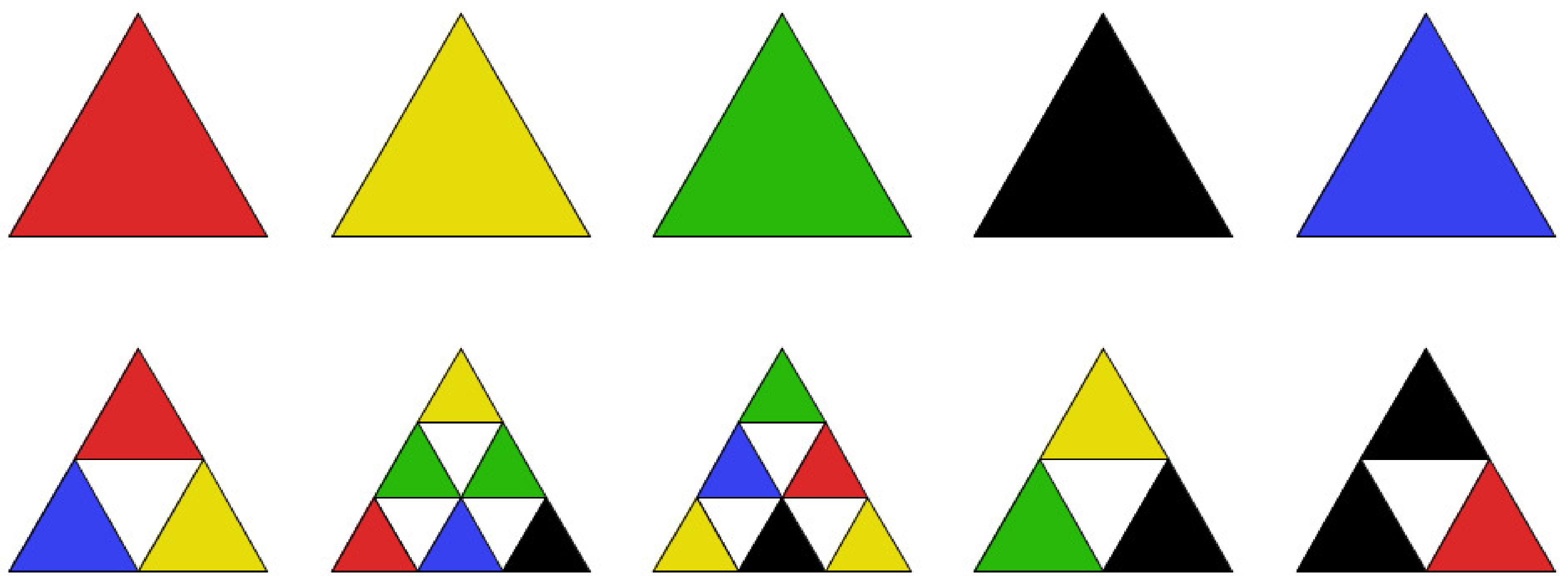
How to get now d_S?

We now need products and sums of ϱ_F and ϱ_G according to the $V-$variable setting.

Define $T_F := \varrho_F M_F$ and $T_G := \varrho_G M_G$, i.e. $T_F = 5$ and $T_G = 90/7$. (Note: These are the mean crossing times through the generating graphs.)

The transformation from level k to level $k+1$ of $V-$tuples of triangles we code up with the help of an $V \times V-$matrix $M^{(k)}(\alpha)$ as follows: (hereby $\alpha > 0$ is a free parameter)

Remember:



$$M^{(k)}(\alpha) = \begin{pmatrix} \left(\frac{1}{T_F}\right)^\alpha & \left(\frac{1}{T_F}\right)^\alpha & 0 & 0 & \left(\frac{1}{T_F}\right)^\alpha \\ \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & 2\left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha \\ \left(\frac{1}{T_G}\right)^\alpha & 2\left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha & \left(\frac{1}{T_G}\right)^\alpha \\ 0 & \left(\frac{1}{T_F}\right)^\alpha & \left(\frac{1}{T_F}\right)^\alpha & \left(\frac{1}{T_F}\right)^\alpha & 0 \\ \left(\frac{1}{T_F}\right)^\alpha & 0 & 0 & 2\left(\frac{1}{T_F}\right)^\alpha & 0 \end{pmatrix}$$

pressure function

$$\gamma_V(\alpha) := \lim_{k\to\infty} \frac{1}{k} \log\left(\frac{1}{V}\left\|M^{(k)}(\alpha)\ldots M^{(1)}(\alpha)\right\|\right),$$

where the norm $\|A\|$ is the sum of all the entries in the matrix A.

Theorem (F+Hambly+Hutchinson, 2010)

$\gamma_V(\alpha)$ is a well defined function of α and independent of the realization of the experiment.(Furstenberg/Kesten 1960).

Moreover it holds that $\gamma_V(.)$ is strictly monotone decreasing and $\exists!\ d : \gamma_V(d) = 0$.

For this zero d of $\gamma_V(.)$ it holds a.s. that $N(x) \sim x^d$.

More precisely, it holds that

$$N(x)x^{-\alpha} \longrightarrow 0 \qquad \mathbf{P}-a.s. \quad \text{for } \alpha > d.$$

and

$$N(x)x^{-\alpha} \longrightarrow \infty \qquad \mathbf{P}-a.s. \quad \text{for } \alpha < d.$$

[FHaHu, 2011] Refinement of the result on the spectral asymptotics:

• For any self similar measure it holds that

$$\lim_{x\to\infty} \frac{\log N(x)}{\log x} = \frac{d_s}{2} \qquad \mathbf{P} - a.s.$$

• In the „flat–measure–case“ it holds that

$$c^{-1}x^{d_s/2}/\phi(x) \le N(x) \le c x^{d_s/2}\phi(x), \quad x \to \infty, \mathbf{P} - a.s,$$

where $\phi(x) := \exp(-c\sqrt{\log x \log\log\log x})$.

• In the „flat–measure–case“ it holds for $V = \infty$ that

$$\lim_{x\to\infty} \frac{N(x)}{x^{d_s/2}} = C, \ \ \mathbf{P} - a.s.$$

[FHaHu, 2011] heat-kernel-estimates (on–diagonal)

In the „flat–measure–case“ exist constants such that $\mathbf{P}$-a.s. for μ-almost every $x \in K$ it holds that

$$c_1 \phi(1/t)^{-c_2} t^{-d_s/2} \le p_t(x,x) \le c_3 \phi(1/t)^{c_4} t^{-d_s/2}, \ \ 0 < t < 1,$$

where $\phi(x) = \exp(-c\sqrt{\log x \log\log\log x})$.

Remark: The measure is not „doubling“!! Existence of heat kernels: [Croydon, 2007]

- Well known results for $V = 1$ are contained now as special cases.

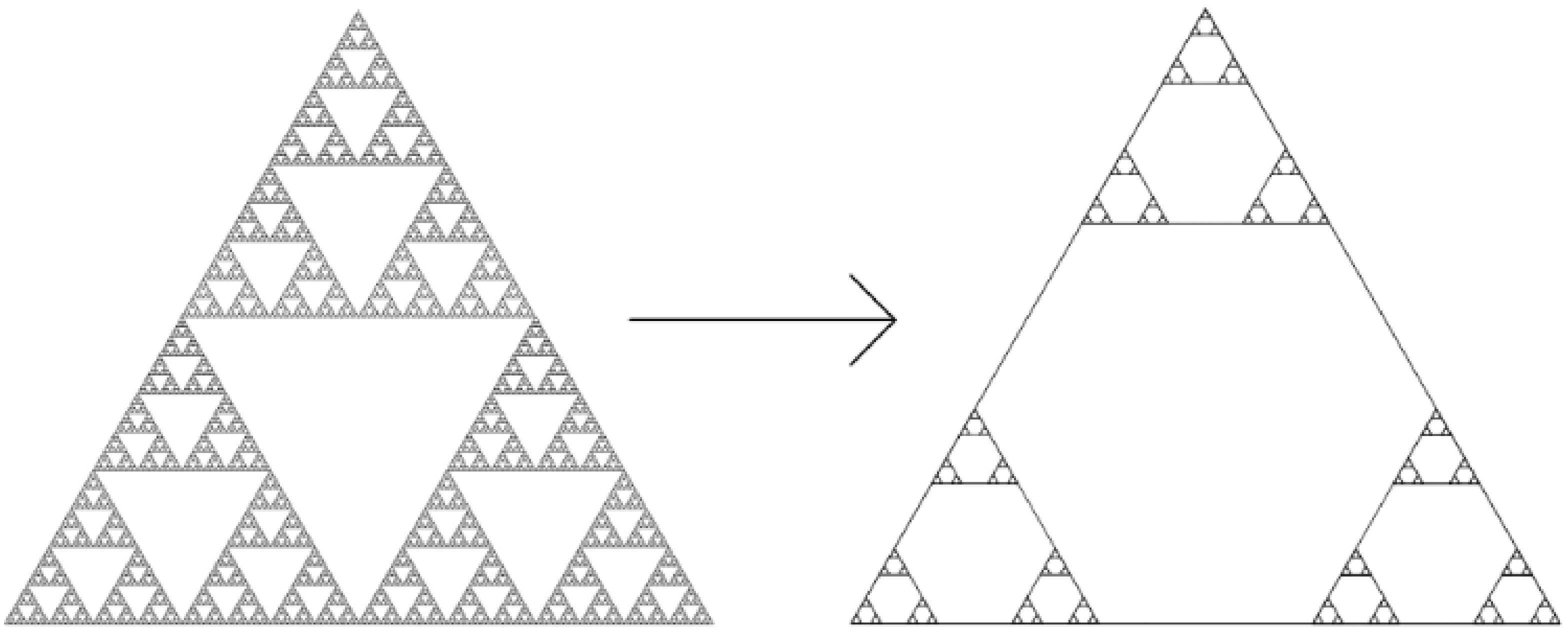
- For $V \geq 2$: no explicite expression for d_S (simulation)

Reference:

U. Freiberg, B.M. Hambly and J.E. Hutchinson
Spectral asymptotics for $V-$variable Sierpinski gaskets.
Ann. Inst. H. Poincare Probab. Stat. 53, 2162–2213, 2017

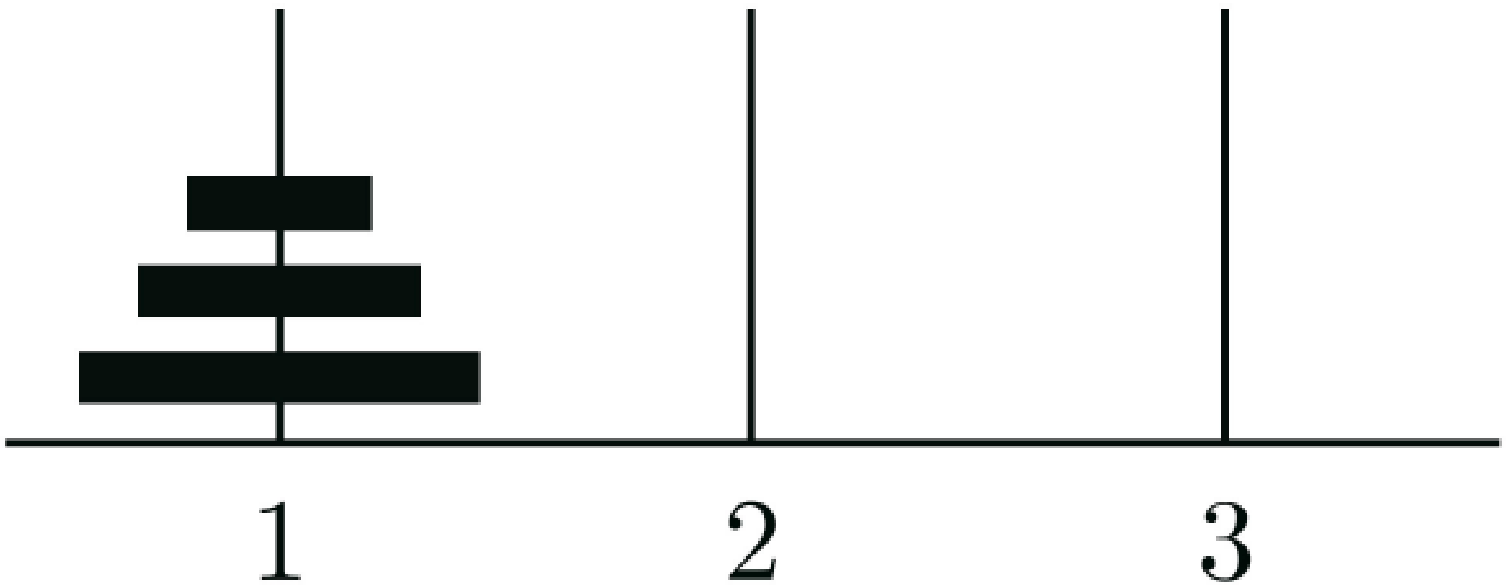
(or, Ben's homepage)

6. Stretched fractals

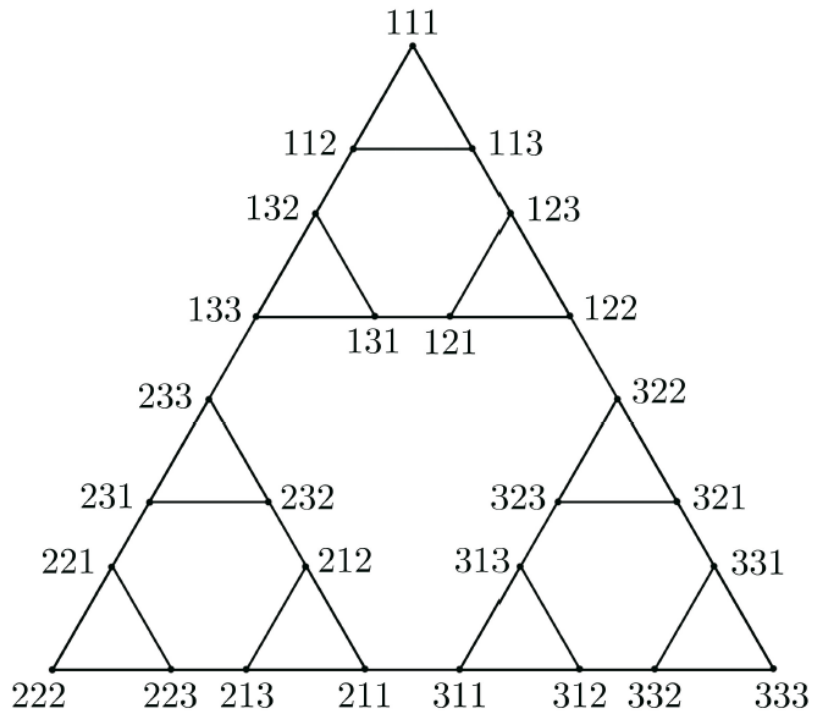


Sierpinski gasket and stretched Sierpinski gasket

Stretched Sierpinski gasket is often called Hanoi graph

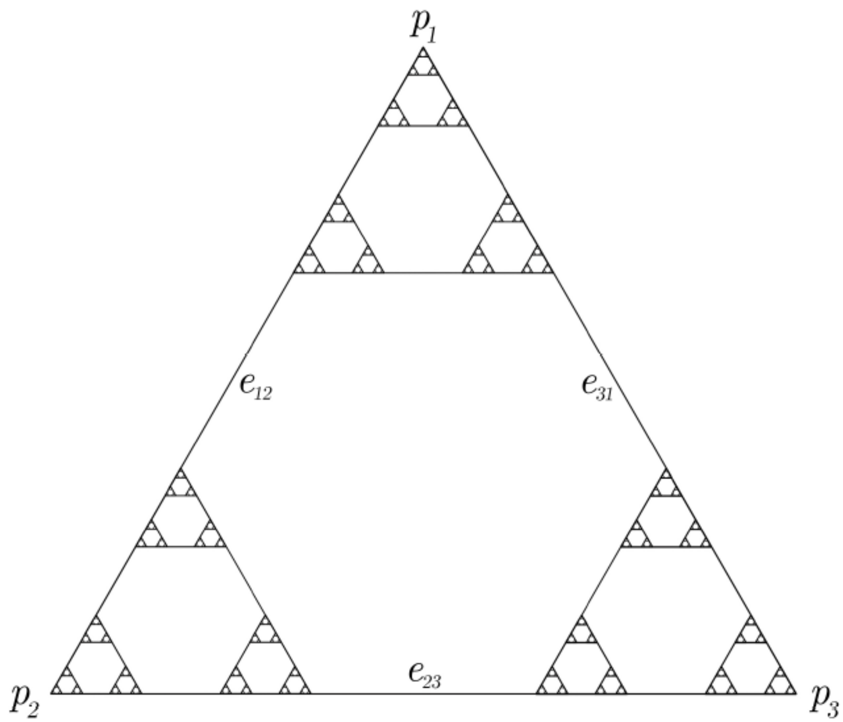


Tower of Hanoi game



Legal moves

see: Hinz, Klavzar, Petr: The tower of Hanoi – myths and maths, 2nd ed. 2018



SSG is an inhomogenous self similar fractal

• Let p_1, p_2, p_3 be the vertex points of an equilateral triangle with side length 1 and for $\alpha \in (0,1)$ define

$$G_i(x) := \frac{1-\alpha}{2}(x - p_i) + p_i, \quad i = 1, 2, 3$$

.

- e_{ij} := line segment between $G_i(p_j)$ and $G_j(p_i)$.

- Then, K_α is the unique compact set with

$$K_\alpha = G_1(K_\alpha) \cup G_2(K_\alpha) \cup G_3(K_\alpha) \cup e_{12} \cup e_{23} \cup e_{31}.$$

- Unique (in the Hausdorff space) solution of

$$\Sigma_\alpha = G_1(\Sigma_\alpha) \cup G_2(\Sigma_\alpha) \cup G_3(\Sigma_\alpha)$$

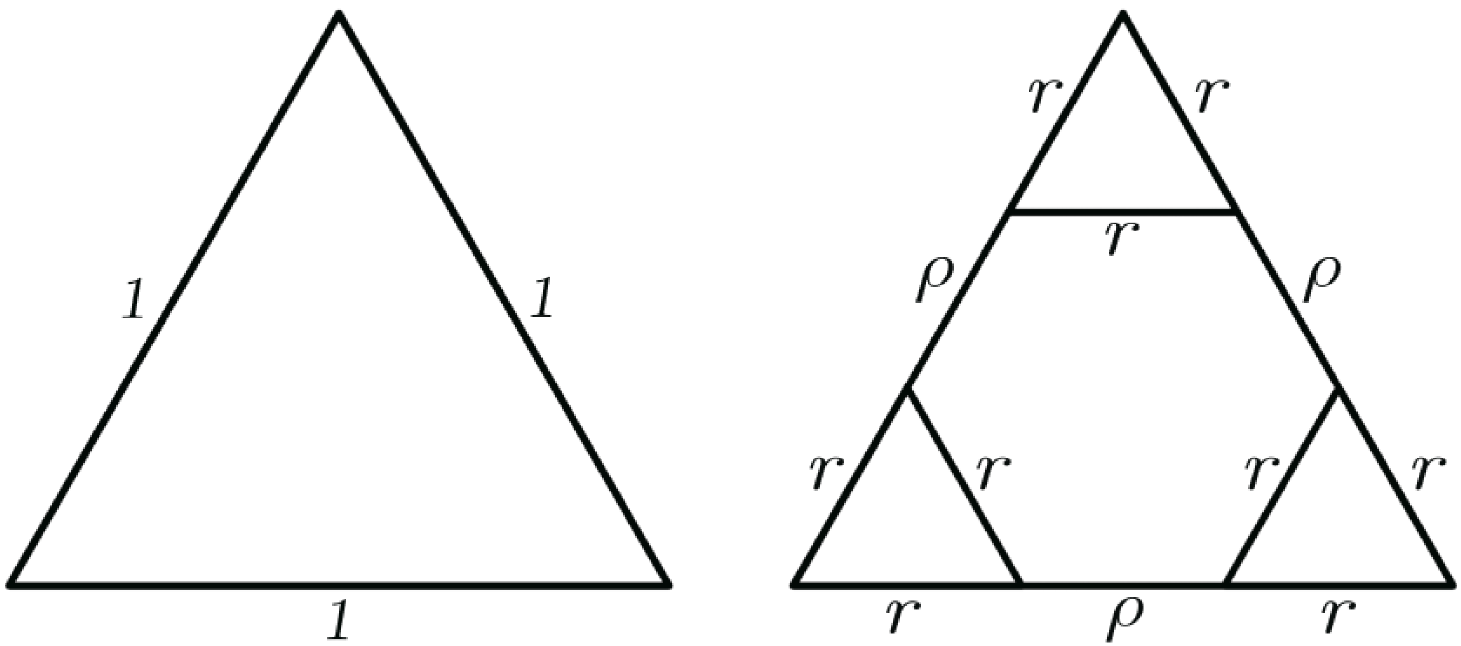
we call fractal part, the rest $K_\alpha \setminus \Sigma_\alpha$ line part.

- Observation: $\alpha \downarrow 0$: $K_\alpha \to K$ Sierpinski gasket in Hausdorff distance, in Hausdorff dimension

(P. Alonso–Ruiz, URF: Hanoi attractors and the Sierpinski gasket, 2012)

- Question: Does the analysis converge? In which sense?

(P. Alonso–Ruiz, URF: Weyl asymptotics for Hanoi attractors, 2017)

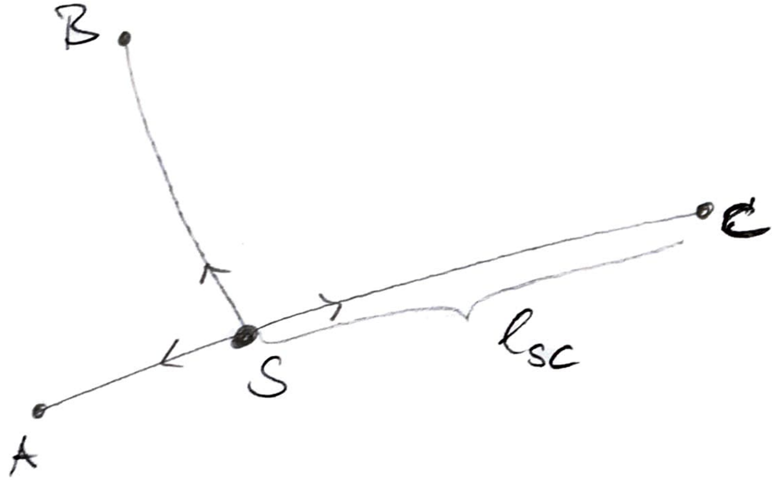


- $\rho + 5/3r = 1$, (ρ, r) matching pair
- more general: $(\rho_n, r_n)_{n\geq 1}$ matching sequence, see: P. Alonso–Ruiz, URF and J. Kigami: Completely symmetric resistance forms on the stretched Sierpinski gasket., J. Fractal Geom. 5 (2018), no. 3, 227–277

• Weyl asymptotics: 3 preprints, on arXiv:
Elias Hauser: Spectral asymptotics on the Hanoi attractor, 2017
Elias Hauser: Oscillations on the Stretched Sierpinski Gasket, 2018.
Elias Hauser: Spectral Asymptotics for Stretched Fractals, 2018.

• many open problems, in particular on the associated stochastic process...

Problem: transition probabilities



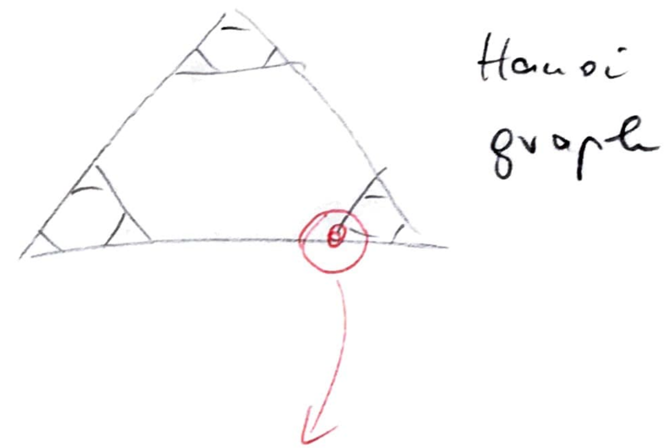
l_{AS}, l_{BS}, l_{CS} ≙ lengths

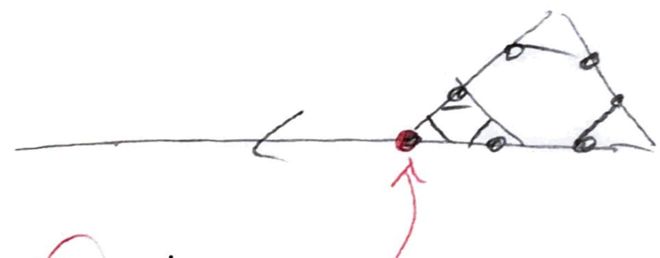
$r_{ij} = l_{ij}$ "resistances"

$c_{ij} := r_{ij}^{-1}$ "conductances"

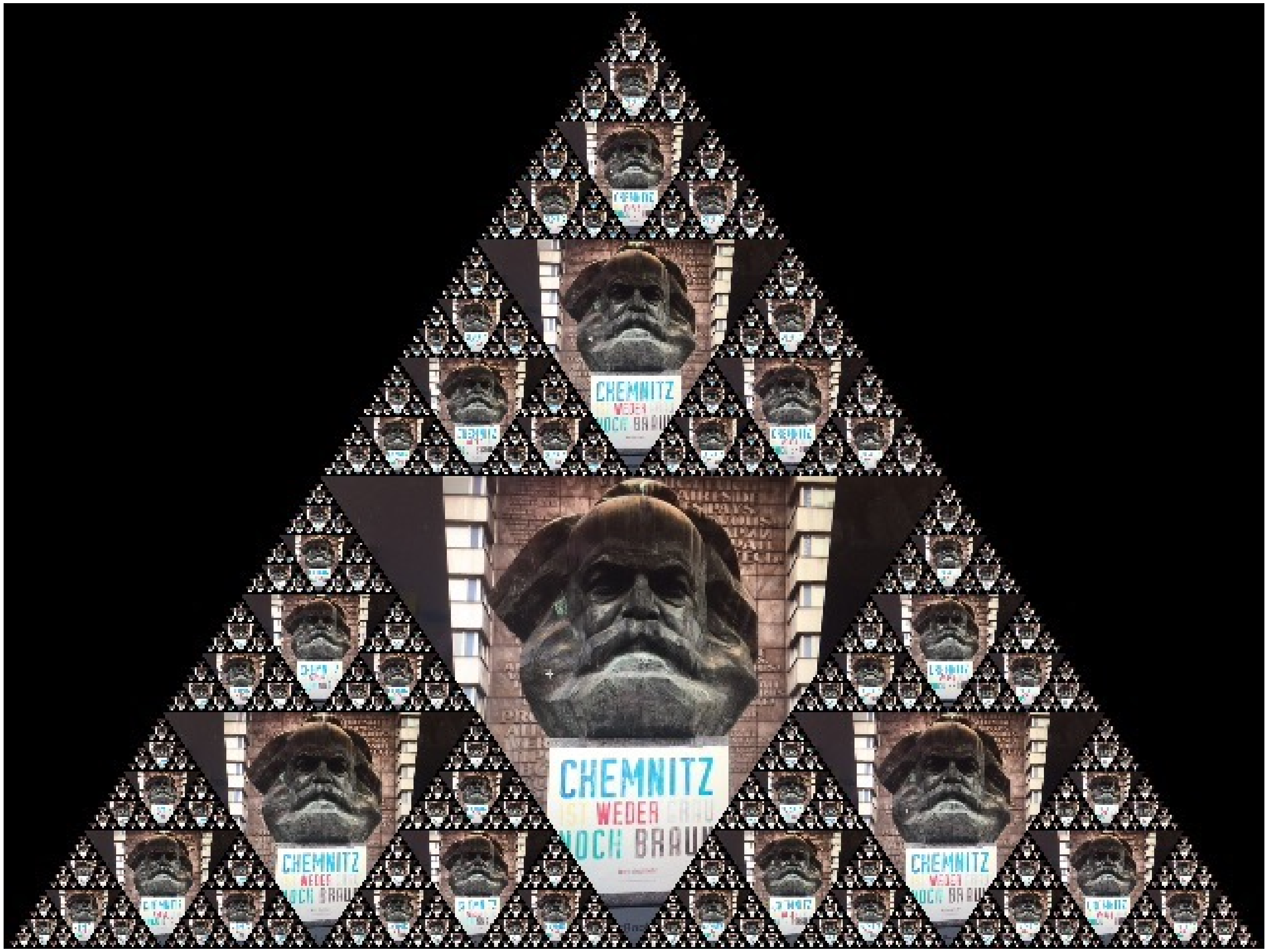
natural:

$$p_{SA} = \frac{c_{SA}}{c_{SA} + c_{SB} + c_{SC}}$$





? transition prob in this point ?



Thank you for your attention!